

# On the monotonicity of the moments of volumes of random simplices

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## Abstract

In a  $d$ -dimensional convex body  $K$  random points  $X_0, \dots, X_d$  are chosen. Their convex hull is a random simplex. The expected volume of a random simplex is monotone under set inclusion, if  $K \subset L$  implies that the expected volume of a random simplex in  $K$  is smaller than the expected volume of a random simplex in  $L$ . Continuing work of Rademacher, it is shown that moments of the volume of random simplices are in general not monotone under set inclusion.

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## 1 Introduction

For a  $d$ -dimensional convex body  $K$ , denote the volume of the convex hull of  $d + 1$  independently uniformly distributed random points  $X_0, \dots, X_d$  in  $K$  by  $V_K$ . Since the  $d + 1$  points are in general position with probability 1, their convex hull is almost surely a full-dimensional simplex. Meckes [4] asked whether the expected volume  $\mathbb{E}V_K$  is monotone under inclusion, i.e. for each pair of convex bodies  $K, L \subseteq \mathbb{R}^d$ ,  $K \subseteq L$  implies

$$\mathbb{E}V_K \leq \mathbb{E}V_L. \quad (1)$$

He also states a weak conjecture concerning the existence of a universal constant  $c > 0$  such that  $K \subseteq L$  implies

$$\mathbb{E}V_K \leq c^d \mathbb{E}V_L.$$

Interest in this question comes from the fact that both conjectures would imply a positive solution to the slicing problem. In fact, the weak conjecture is an equivalent formulation, see e.g. [6]. For a more general statement of the conjecture we refer to [8].

In this paper, we investigate the question for arbitrary moments of the volumes of random simplices. Let  $K, L \subseteq \mathbb{R}^d$  denote convex bodies. For  $d \in \mathbb{N}$ , define  $k_d$  as the critical exponent such that

- (i)  $K \subseteq L$  implies  $\mathbb{E}V_K^k \leq \mathbb{E}V_L^k$  for all  $k < k_d$ , and
- (ii) there exist  $K \subseteq L$  with  $\mathbb{E}V_K^k > \mathbb{E}V_L^k$  for all  $k \geq k_d$ .

At first it is unclear whether there is a critical exponent at all, where the behaviour switches from monotonicity to the existence of counterexamples precisely at  $k_d$ . This issue will be a byproduct of the following results. And the interesting second question is whether  $k_d < \infty$ .

In 2012, Rademacher [6] showed that Meckes' stronger conjecture (1) is not true in general. (Note that the problem is trivial in dimension 1 where  $k_1 = \infty$  and monotonicity holds.) More precisely, Rademacher proved the following, where the main point is the surprising existence of counterexamples, i.e.  $k_d < \infty$ :

**Theorem 1.** *In the planar case,  $3 \leq k_2 < \infty$ , and in dimension three,  $1 \leq k_3 < \infty$  holds. In higher dimension,  $k_d = 1$  holds for all  $d \geq 4$ .*

Our main theorem computes the constant  $k_2$  and makes  $k_3$  more precise.

**Theorem 2.** *In the planar case,  $k_2 = 3$  holds. In dimension three,  $k_3 \in \{1, 2\}$ .*

The reader might recognize that there is still one open task, namely to prove  $k_3 = 1$ , i.e. to disprove monotonicity of the expected volume of a random tetrahedron in dimension three. Numerical simulations show that there is a counterexample (as already conjectured by Rademacher), but a rigorous proof is still missing.

Since a direct proof of this issue is somewhat involved, one may use a very crucial lemma, stated here. In the following, we denote by  $V_{K,x}$  the volume of a random simplex, which is the convex hull of a fixed point  $x$  and  $d$  independent uniform random points in  $K$ .

**Lemma 1** (Rademacher [6]). *For  $k, d \in \mathbb{N}$ , monotonicity under inclusion of the map*

$$K \mapsto \mathbb{E}V_K^k,$$

*where  $K$  ranges over all  $d$ -dimensional convex bodies, holds if and only if we have for each convex body  $K \subseteq \mathbb{R}^d$  and for each  $x \in \text{bd } K$  that*

$$\mathbb{E}V_K^k \leq \mathbb{E}V_{K,x}^k.$$

The lemma allows us to consider one convex body  $K$ , rather than a pair of convex bodies, and compute two different moments: the moment of the volume of a random simplex in  $K$  as well as the same, but fixing one of the  $d+1$  points to be a point on the boundary of  $K$ , denoted by  $\text{bd } K$ .

Rademacher takes  $K$  to be a  $d$ -dimensional halfball and  $x$  is the midpoint of the base which is a  $(d-1)$ -dimensional ball. These form

the counterexamples to the monotonicity in Theorem 1 for all moments for  $d \geq 4$ , and to the monotonicity of all but finitely many moments for  $d = 2, 3$ .

In the background of our Theorem 2, there are a more detailed computation of Rademacher's counterexample in dimension 3 and the construction of a new counterexample in dimension 2. Here we have to compute the area of a random triangle in a triangle where one vertex is fixed at the midpoint of one edge.

**Theorem 3.** *Let  $T \subseteq \mathbb{R}^2$  be a triangle and  $x$  the midpoint of an edge of  $T$ . Then we have for the  $k$ -th moment of  $V_{T,x} = \text{vol conv}(x, X_1, X_2)$ :*

$$\frac{\mathbb{E}V_{T,x}^k}{\text{vol } T^k} = \frac{2^{3-k}}{(k+1)(k+2)^2(k+3)} \left( \sum_{l=1}^{k+1} \binom{k+2}{l}^{-1} + 1 \right).$$

Coming back to the (lack of) monotonicity of the expected volume of a random tetrahedron in dimension 3, it has already been conjectured by Rademacher that the above example, the halfball  $B_3^+$  together with one point at the origin  $o$  of its base, should also form a counterexample in this case. The value  $\mathbb{E}V_{B_3^+,o}$  is known (and will be given in Section 2), but for  $\mathbb{E}V_{B_3^+}$ , the precise value is an open task. Numerical computations show that

$$0.028105 \approx \mathbb{E}V_{B_3^+} > \mathbb{E}V_{B_3^+,o} = \frac{9\pi}{1024} = 0.0276 \dots$$

A second counterexample is given by a tetrahedron  $T$  and  $x$  the centroid of one of its facets. Here  $\mathbb{E}V_T$  is known, but  $\mathbb{E}V_{T,x}$  is missing. Again, by numerical integration we obtain

$$0.0173 \dots = \frac{13}{720} - \frac{\pi^2}{15015} = \mathbb{E}V_T > \mathbb{E}V_{T,x} \approx 0.015901.$$

This paper is organized in the following way. In Section 2, we give some auxiliary results and notation. Then we compute the moments of the area of a random triangle inside a triangle. This result will be used in Section 4 for the proof of the main theorem, which is an extension of two theorems of Rademacher.

As a general reference for the tools and results we need in the following, we refer to the book on Stochastic and Integral Geometry by Schneider and Weil [9]. More recent surveys on random polytopes are due to Hug [3] and Reitzner [8].

## 2 Preliminaries

In the following, we need some well-known results on random polytopes which we collect here for later use. We start with dimension one, where

a convex set is an interval  $I$  and the volume  $V_I$  of a random simplex is the distance between two random points.

**Lemma 2** (cf., e.g., [10]). *Assume  $I$  is an interval of length  $l$ . Then*

$$\mathbb{E}V_I^k = \frac{2l^k}{(k+1)(k+2)}.$$

The only convex body for which the moments of random simplices are known in all dimensions is the unit ball  $B_d$ . They were computed by Miles [5]. Let  $\kappa_d = \text{vol } B_d = \pi^{d/2}/\Gamma(1+d/2)$  be the volume of  $B_d$ , where  $\Gamma(\cdot)$  denotes the gamma function, and let  $\omega_d = \text{vol } S_{d-1} = d\kappa_d$  be the  $(d-1)$ -dimensional volume of the boundary of  $B_d$ .

**Theorem 4** (cf. [5] or [9], Theorem 8.2.3). *For any  $d, k \in \mathbb{N}$ , we have*

$$\mathbb{E}V_{B_d}^k = \frac{1}{(d!)^k} \left( \frac{\kappa_{d+k}}{\kappa_d} \right)^{d+1} \frac{\kappa_{d(d+k+1)}}{\kappa_{(d+1)(d+k)}} \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

Observe that these values coincide with those from Lemma 2 for  $d = 1$  and  $l = 2$ . It was proved by Blaschke (for  $d = 3$ ) and Groemer (for arbitrary  $d$ ) that these values are extremal in the sense that under all convex bodies of volume one,  $\mathbb{E}V_K^k$  is minimized for the ball.

**Theorem 5** (Blaschke-Groemer, cf. [9], Theorem 8.6.3). *Let  $d, k \in \mathbb{N}$ . Among all  $d$ -dimensional convex bodies, the map*

$$K \mapsto \frac{\mathbb{E}V_K^k}{\text{vol } K^k}$$

*attains its minimum if and only if  $K$  is an ellipsoid.*

In the following, we need the expected volume of a random simplex where one point is fixed at the origin and the others are uniformly chosen in the unit ball. Again, this result is due to Miles.

**Theorem 6** (cf. [5] or [9], Theorem 8.2.2). *For any  $d, k \in \mathbb{N}$ , we have*

$$\mathbb{E}V_{B_d, o}^k = \frac{1}{(d!)^k} \left( \frac{\kappa_{d+k}}{\kappa_d} \right)^d \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

Finally, we prove here a general lemma that has already been used by Rademacher [6] in the case of the unit ball. It seems to be well-known, but we could not find a rigorous proof in the literature.

**Lemma 3.** *Assume  $K$  is a  $d$ -dimensional convex body which is symmetric with respect to  $x \in K$  and let  $H^+$  be a half-space containing  $x$  on its boundary. Then*

$$\mathbb{E}V_{K \cap H^+, x}^k = \mathbb{E}V_{K, x}^k.$$

*Proof.* Without loss of generality, we identify  $x$  with the origin  $o$ . Furthermore, we denote the intersection of  $K$  with the half-space  $H^+$  by  $K^+$ . Since the volume of the simplex  $\text{conv}(o, x_1, \dots, x_d)$  is just the absolute value of the determinant of the matrix containing the vectors  $x_i$ , divided by  $d!$ , it holds:

$$\mathbb{E}V_{K^+, o}^k = \frac{1}{d!} \mathbb{E}_{X_i \in K^+} |\det(X_1, \dots, X_d)|^k.$$

Assume that  $\epsilon_i \in \{\pm 1\}$  for  $i = 1, \dots, d$ . Because the absolute value of the determinant is an even function,  $|\det(X_1, \dots, X_d)| = |\det(\epsilon_1 X_1, \dots, \epsilon_d X_d)|$  for any choice of  $\epsilon_i$ . It follows, summing over all possible combinations of signs,

$$\begin{aligned} \mathbb{E}V_{K^+, o}^k &= \frac{1}{2^d d!} \sum_{\epsilon_i = \pm 1} \mathbb{E}_{X_i \in K^+} |\det(\epsilon_1 X_1, \dots, \epsilon_d X_d)|^k \\ &= \frac{1}{2^d d!} \frac{1}{(\text{vol } K^+)^d} \sum_{\epsilon_i = \pm 1} \int_{(K^+)^d} |\det(\epsilon_1 x_1, \dots, \epsilon_d x_d)|^k d(x_1, \dots, x_d). \end{aligned}$$

Since the reflection of each point in  $K^+$  lies in  $K \setminus K^+$ , in fact we integrate over all  $d$ -tuples of points lying in  $K$ . Because, due to symmetry, the volume of  $K^+$  is just half of the volume of  $K$ , we get

$$\begin{aligned} \mathbb{E}V_{K^+, o}^k &= \frac{1}{2^d d!} \frac{1}{(\text{vol } K^+)^d} \int_{K^d} |\det(x_1, \dots, x_d)|^k d(x_1, \dots, x_d) \\ &= \frac{1}{d!} \mathbb{E}_{X_i \in K} |\det(X_1, \dots, X_d)|^k = \mathbb{E}V_{K, o}^k. \end{aligned}$$

□

Let  $H^+$  be any halfspace containing the origin in its boundary. Denote by  $B_d^+ = B_d \cap H^+$  half of the  $d$ -dimensional unit ball. Using the lemma above, we immediately see that for any  $d, k \in \mathbb{N}$  we have

$$\mathbb{E}V_{B_d^+, o}^k = \frac{1}{(d!)^k} \left( \frac{\kappa_{d+k}}{\kappa_d} \right)^d \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}. \quad (2)$$

For evaluation of the occurring expressions — and in particular of the volume of the unit ball —, the following estimates are useful.

**Lemma 4** (Borgwardt [2], also cf. [6]). *For  $d \geq 2$ , we have*

$$\sqrt{\frac{d}{2\pi}} \leq \frac{\kappa_{d-1}}{\kappa_d} \leq \sqrt{\frac{d+1}{2\pi}}.$$

### 3 Random triangles in a triangle

Essential for our investigations in the planar case is the expected area of a random triangle in a given triangle  $T$ . In particular, we need the expected area in the case where one point is fixed at the midpoint of an edge. This is the statement of Theorem 3, which is proved in this section.

Let  $x$  be the midpoint of an edge of  $T$ . We show that

$$\frac{\mathbb{E}V_{T,x}^k}{\text{vol } T^k} = \frac{2^{3-k}}{(k+1)(k+2)^2(k+3)} \left( \sum_{l=1}^{k+1} \binom{k+2}{l}^{-1} + 1 \right).$$

*Proof.* Since the moments of the volume of the random triangle do not depend on the shape of the triangle, we can consider the specific triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x + y \leq 1\},$$

i.e. the triangle with vertices  $E_0 = (0, 0)$ ,  $E_1 = (1, 0)$  and  $E_2 = (0, 1)$ . Note that its area is  $\text{vol } T = 1/2$ . We choose  $x = (1/2, 1/2)$ , the midpoint of the edge  $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x + y = 1\}$ , to be the fixed vertex of the random triangle.

Using the affine Blaschke-Petkantschin formula — see e.g. [9] —, we transform our integral and integrate over all lines  $H$  intersecting the triangle.

$$\begin{aligned} \mathbb{E}V_{T,x}^k &= \frac{1}{\text{vol } T^2} \int_{T^2} \text{vol conv}(x, x_1, x_2)^k d(x_1, x_2) \\ &= 4 \int_{A(2,1)} \int_{(H \cap T)^2} \text{vol conv}(x, x_1, x_2)^k \|x_1 - x_2\| d(x_1, x_2) dH, \end{aligned}$$

where  $A(2, 1)$  denotes the affine Grassmannian of lines in  $\mathbb{R}^2$ . We represent a line  $H$  by its unit normal vector  $u \in S_1$  and its distance  $t > 0$  from the origin and we therefore denote the line by

$$H_{t,u} = \{x \in \mathbb{R}^2 : \langle x, u \rangle = t\}.$$

We choose the normalization of the Haar measure  $dH$  in such a way that  $dH = dt du$ , where  $dt$  and  $du$  correspond to Lebesgue measures in  $\mathbb{R}$  and  $S_1$ . The area of the triangle  $\text{conv}(x, x_1, x_2)$  is the product of the length  $\|x_1 - x_2\|$  of its base and its height  $d(H_{t,u}, x)$ , divided by

2. We write the appearing integral as an expectation to get

$$\begin{aligned} \mathbb{E}V_{T,x}^k &= 4 \int_{S_1} \int_0^\infty \frac{d(H_{t,u}, x)^k}{2^k} \int_{(H_{t,u} \cap T)^2} \|x_1 - x_2\|^{k+1} d(x_1, x_2) dt du \\ &= 4 \int_{S_1} \int_0^\infty \frac{d(H_{t,u}, x)^k}{2^k} \text{vol}(H_{t,u} \cap T)^2 \mathbb{E}V_{H_{t,u} \cap T}^{k+1} dt du. \end{aligned}$$

The  $(k+1)$ -st moment of the distance of two random points in the intersection  $H_{t,u} \cap T$  has already been given in Lemma 2. Hence, we obtain

$$\mathbb{E}V_{T,x}^k = \frac{2^{3-k}}{(k+2)(k+3)} \int_{S_1} \int_0^\infty d(H_{t,u}, x)^k \text{vol}(H_{t,u} \cap T)^{k+3} dt du.$$

A line  $H_{t,u}$  that intersects the triangle  $T$  a.s. meets exactly two edges of  $T$ . It splits  $T$  into a triangle and a quadrangle. We say that  $H_{t,u}$  cuts off the vertex  $E_i$  from  $T$  if  $E_i$  is contained in the triangular part. Furthermore, we write

$$\mathcal{I}^{(i)} = \int_{S_1} \int_0^\infty \mathbf{1}(H_{t,u} \text{ cuts off } E_i \text{ from } T) d(H_{t,u}, x)^k \text{vol}(H_{t,u} \cap T)^{k+3} dt du$$

for  $i = 0, 1, 2$ , which gives

$$\mathbb{E}V_{T,x}^k = \frac{2^{3-k}}{(k+2)(k+3)} \left( \mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} \right).$$

We state the following lemma which will be proved right after the end of the proof of this proposition:

**Lemma 5.** *It holds:*

$$\begin{aligned} (i) \quad \mathcal{I}^{(0)} &= \frac{1}{2^k(k+1)(k+2)} \sum_{l=1}^{k+1} \binom{k+2}{l}^{-1}, \\ (ii) \quad \mathcal{I}^{(1)} = \mathcal{I}^{(2)} &= \frac{1}{2^{k+1}(k+1)(k+2)}. \end{aligned}$$

Utilizing this lemma, we get

$$\mathbb{E}V_{T,x}^k = \frac{2^{3-2k}}{(k+1)(k+2)^2(k+3)} \left( \sum_{l=1}^{k+1} \binom{k+2}{l}^{-1} + 1 \right).$$

□

*Proof of Lemma 5.* We start with the computation of  $\mathcal{I}^{(0)}$ . We substitute  $z = u/t$  and get with  $H_z = \{x \in \mathbb{R}^2 : \langle x, z \rangle = 1\}$  and  $dt du = |z|^{-3} dz$  that

$$\mathcal{I}^{(0)} = \int_{\mathbb{R}^2} \mathbb{1}(H_z \text{ cuts off } E_0 \text{ from } T) d(H_z, x)^k \text{vol}(H_z \cap T)^{k+3} |z|^{-3} dz.$$

With a second substitution by  $a = z_1^{-1}, b = z_2^{-1}$ , we get by  $a$  the abscissa of the point of intersection of a line with the  $x$ -axis and by  $b$  the ordinate of the intersection of the line with the  $y$ -axis. We write  $H_{a,b}$  for the line represented by the parameters  $a$  and  $b$  and have  $H_{a,b} = \{x \in \mathbb{R}^2 : \langle x, (a^{-1}, b^{-1}) \rangle = 1\}$  and  $dz = da db / (a^2 b^2)$ . Considering the appearing indicator function, we see that a line  $H_{a,b}$  cuts off  $E_0$  from  $T$  — or, in other words, intersects both catheti of  $T$  — if and only if  $a$  and  $b$  both lie between 0 and 1. Our integral consequently transforms into

$$\mathcal{I}^{(0)} = \int_0^1 \int_0^1 \frac{d(H_{a,b}, x)^k \text{vol}(H_{a,b} \cap T)^{k+3}}{(a^2 + b^2)^{3/2}} ab da db.$$

We use the notation  $l(a, b) = \text{vol}(H_{a,b} \cap T)$  for the length of the intersection of  $T$  with a line  $H_{a,b}$ , and  $h(a, b) = d(H_{a,b}, x)$  for the distance of this line from  $x$ . We get by easy computations that

$$l(a, b) = \sqrt{a^2 + b^2} \quad \text{and} \quad h(a, b) = (a + b - 2ab) / (2\sqrt{a^2 + b^2}).$$

which yields

$$\begin{aligned} \mathcal{I}^{(0)} &= \int_0^1 \int_0^1 \frac{l(a, b)^{k+3} h(a, b)^k}{(a^2 + b^2)^{3/2}} ab da db = \frac{1}{2^k} \int_0^1 \int_0^1 (a + b - 2ab)^k ab da db \\ &= \frac{1}{2^k} \int_0^1 \int_0^1 ((a(1-b) + b(1-a)))^k ab da db \\ &= \frac{1}{2^k} \sum_{l=0}^k \binom{k}{l} \int_0^1 \int_0^1 a^{l+1} (1-a)^{k-l} b^{k-l+1} (1-b)^l da db \\ &= \frac{1}{2^k} \sum_{l=0}^k \binom{k}{l} \frac{(l+1)!(k-l)!}{(k+2)!} \frac{l!(k-l+1)!}{(k+2)!} \\ &= \frac{1}{2^k(k+1)(k+2)} \sum_{l=1}^{k+1} \binom{k+2}{l}^{-1}, \end{aligned}$$

and we arrive at the expression stated in (i).



Considering statement (ii), we first note that  $\mathcal{I}^{(1)} = \mathcal{I}^{(2)}$  due to symmetry. Hence it suffices to compute  $\mathcal{I}^{(1)}$ . Furthermore, the integrals are affine invariant and we can transform the triangle into a similar one, bringing the point  $x$  to the midpoint  $(1/2, 0)$  of the edge on the  $x$ -axis, and exchanging the vertices  $E_i$  clockwise. Now, a line cutting off  $E_1$  from  $T$  intersects the triangle in both catheti and  $a$  and  $b$  lie between 0 and 1.

The function  $h(a, b)$  changes its sign at  $a = 1/2$ . As before, we get again  $l(a, b) = \sqrt{a^2 + b^2}$ , and by another straightforward computation,

$$h(a, b) = \begin{cases} \frac{b-2ab}{2\sqrt{a^2+b^2}} & \text{for } 0 \leq a \leq \frac{1}{2}, \\ \frac{2ab-b}{2\sqrt{a^2+b^2}} & \text{for } \frac{1}{2} \leq a \leq 1. \end{cases}$$

The occuring double integral can be solved by partial integration.

$$\begin{aligned} \int_0^1 \int_0^{1/2} \frac{l(a, b)^{k+3} h(a, b)^k}{(a^2 + b^2)^{3/2}} ab \, da \, db &= \frac{1}{2^k} \int_0^1 \int_0^{1/2} (b - 2ab)^k ab \, da \, db \\ &= \frac{1}{2^{k+2}(k+1)(k+2)} \int_0^1 b^{k+1} \, db = \frac{1}{2^{k+2}(k+1)(k+2)^2}, \end{aligned}$$

and analogously

$$\int_0^1 \int_{1/2}^1 \frac{l(a, b)^{k+3} h(a, b)^k}{(a^2 + b^2)^{3/2}/(ab)} \, da \, db = \frac{2k+3}{2^{k+2}(k+1)(k+2)^2}.$$

A combination of both results yields the proof of statement (2).  $\square$

## 4 Proof of Theorem 2

For the first step of the proof, we refine Rademacher's method of proof of Theorem 1.

### 4.1 Random polytopes in hemispheres

Let  $B_d^+$  be half of the  $d$ -dimensional unit ball and  $L$  the ball of volume  $\text{vol } B_d^+$ . According to Theorem 5,

$$\mathbb{E}V_{B_d^+}^k > \mathbb{E}V_L^k,$$

and since  $\mathbb{E}V_L^k = 2^{-k} \mathbb{E}V_{B_d^+}^k$ , Theorem 4 implies

$$\mathbb{E}V_{B_d^+}^k > \frac{1}{2^k (d!)^k} \left( \frac{\kappa_{d+k}}{\kappa_d} \right)^{d+1} \frac{\kappa_{d(d+k+1)}}{\kappa_{(d+1)(d+k)}} \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

On the other hand, by equation (2),

$$\mathbb{E}V_{B_d^+, o}^k = \frac{1}{(d!)^k} \left( \frac{\kappa_{d+k}}{\kappa_d} \right)^d \frac{\omega_1 \cdots \omega_k}{\omega_{d+1} \cdots \omega_{d+k}}.$$

Combining these statements, we get

$$\frac{\mathbb{E}V_{B_d^+, o}^k}{\mathbb{E}V_{B_d^+}^k} < 2^k \frac{\kappa_d}{\kappa_{d+k}} \frac{\kappa_{(d+1)(d+k)}}{\kappa_{d(d+k+1)}}. \quad (3)$$

Lemma 4 and the inequality  $a/b \leq (a+1)/(b+1)$  for  $0 \leq a \leq b$  yield

$$\frac{\mathbb{E}V_{B_d^+, o}^k}{\mathbb{E}V_{B_d^+}^k} < 2^k \left( \frac{(d+2) \cdots (d+k+1)}{(d(d+k+1)+1) \cdots (d(d+k+1)+k)} \right)^{\frac{1}{2}} = q(d, k)^{\frac{1}{2}},$$

which is Equation (5) in [6]. Now consider the series

$$q(2, k) = 4^k \frac{4 \cdots (k+3)}{(2(k+3)+1) \cdots (2(k+3)+k)}, \quad k \in \mathbb{N},$$

which is strictly decreasing for  $k \geq 4$ . This can be shown by a computation of the ratio

$$\frac{q(2, k+1)}{q(2, k)} = \frac{4(k+4)(2k+7)(2k+8)}{(3k+7)(3k+8)(3k+9)}.$$

Furthermore,  $q(2, k)$  is smaller than 1 for  $k = 11$ , and therefore the same is true for  $k \geq 11$ . The values  $k = 3, \dots, 10$  remain open and will be discussed in the next subsection.

In order to solve the question in dimension  $d = 3$ , we investigate

$$\frac{q(3, k+1)}{q(3, k)} = \frac{4(k+5)(3k+13) \cdots (3k+15)}{(4k+13) \cdots (4k+16)}.$$

Again, it can be shown easily that this series is strictly decreasing for  $k \geq 2$ , and  $q(3, k)$  is smaller than 1 for  $k = 4$  and therefore also for  $k \geq 4$ . For  $k = 2, 3$  we directly investigate the fraction

$$2^k \frac{\kappa_3}{\kappa_{k+3}} \frac{\kappa_{4(k+3)}}{\kappa_{3(k+4)}}$$

in formula 3 and obtain that this equals 1 for  $k = 2$  and gives for  $k = 3$ :

$$2^3 \frac{\kappa_3}{\kappa_6} \frac{\kappa_{24}}{\kappa_{21}} = 0,384 \dots < 1,$$

yielding the result.  $\square$

## 4.2 Random polygons in a triangle

It remains to give counterexamples for  $d = 2$  in the cases  $k = 3, \dots, 10$ . Here we prove that random triangles in triangles are suitable for our purposes by giving the explicit values.

According to Reed [7] und Alagar [1], it holds for a triangle  $T$  of volume one that

$$\mathbb{E}V_T^k = \frac{12}{(k+1)^3(k+2)^3(k+3)(2k+5)} \times \left( 6(k+1)^2 + (k+2)^2 \sum_{i=0}^k \binom{k}{i}^{-2} \right).$$

Using Proposition 3 and recalling that  $x$  is the midpoint of an edge of  $T$ , we have

$$\frac{\mathbb{E}V_{T,x}^k}{\mathbb{E}V_T^k} = \frac{(k+1)^2(k+2)(2k+5)}{3 \cdot 2^{k-1}} \frac{\sum_{l=1}^{k+1} \binom{k+2}{l}^{-1} + 1}{6(k+1)^2 + (k+2)^2 \sum_{i=0}^k \binom{k}{i}^{-2}}.$$

We evaluate this expression for  $k = 3, \dots, 10$ .

| $k$ | $\mathbb{E}V_{T,x}^k$ | $\mathbb{E}V_T^k$ | $\mathbb{E}V_{T,x}^k/\mathbb{E}V_T^k$ |
|-----|-----------------------|-------------------|---------------------------------------|
| 3   | 1/375                 | 31/9 000          | 24/31 $\approx$ 0.774194              |
| 4   | 13/21 600             | 1/900             | 13/24 $\approx$ 0.541667              |
| 5   | 151/987 840           | 1 063/2 469 600   | 755/2 126 $\approx$ 0.355127          |
| 6   | 1/23 520              | 403/2 116 800     | 90/403 $\approx$ 0.223325             |
| 7   | 83/6 531 840          | 211/2 268 000     | 2 075/15 192 $\approx$ 0.136585       |
| 8   | 73/18 144 000         | 13/264 600        | 511/6 240 $\approx$ 0.081891          |
| 9   | 1 433/1073318400      | 2 593/93 915 360  | 10 031/207 440 $\approx$ 0.0483562    |
| 10  | 647/1 405 071 360     | 697/42 688 800    | 22 645/802 944 $\approx$ 0.0282025    |

Table 1:  $\mathbb{E}V_{T,x}^k$  and  $\mathbb{E}V_T^k$  for a triangle  $T$  of volume 1

Note that for  $k = 2$ , we have  $\mathbb{E}V_{T,x}^2/\mathbb{E}V_T^2 = 1$ .

□

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